Pulling Two Correlated Normally-Distributed Random Variates
Part II - The Gaussian Copula and Negative Correlation

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Note: Part II (negative correlation) should be read in conjunction with Part I (positive correlation) as all definitions in Part I apply to Part II.

Part I defined two random variates $\tilde{X}$ and $\tilde{Y}$ that are pulled from the distribution of X and the distribution of Y, respectively, and have a positive pairwise correlation of ρ_{xy}. These random variates were defined as follows...

$$\tilde{X} = \left[\sqrt{\rho_{xy}}\, Z_c + \sqrt{1-\rho_{xy}}\, Z_x\right]\sigma_x + \mu_x \tag{1}$$

$$\tilde{Y} = \left[\sqrt{\rho_{xy}}\, Z_c + \sqrt{1-\rho_{xy}}\, Z_y\right]\sigma_y + \mu_y \tag{2}$$

To pull normally-distributed random variates that have a negative correlation we will make a few changes to the two equations above. The first thing that we will do is define a new correlation coefficient $\hat{\rho}_{xy}$ to be the absolute value of the actual negative pairwise correlation coefficient ρ_{xy}. In equation form this definition is...

$$\hat{\rho}_{xy} = \left|\rho_{xy}\right| \tag{3}$$

We will now adjust equations (1) and (2) for negative correlation. The equations for the random variates with negative correlation become...

$$\tilde{X} = \left[\sqrt{\hat{\rho}_{xy}}\, Z_c + \sqrt{1-\hat{\rho}_{xy}}\, Z_x\right]\sigma_x + \mu_x \tag{4}$$

$$\tilde{Y} = -\left[\sqrt{\hat{\rho}_{xy}}\, Z_c + \sqrt{1-\hat{\rho}_{xy}}\, Z_y\right]\sigma_y + \mu_y \tag{5}$$

In the following section we will prove that the mean and variance of $\tilde{X}$ and $\tilde{Y}$ remain unchanged from Part I. We will also prove that the pairwise correlation between $\tilde{X}$ and $\tilde{Y}$ is $-\hat{\rho}_{xy}$.

Proofs

Since equation (4) did not change from Part I the mean and variance of $\tilde{X}$ also did not change. The mean and variance of $\tilde{X}$ are...

$$mean = \mu_x \;\; ...and... \;\; variance = \sigma_x^2 \tag{6}$$

The mean of $\tilde{Y}$ is equal to μ_y, which also did not change from Part I. This proof requires the result of the expectation calculated in Appendix equation A.

$$mean = \mathbb{E}\left[\tilde{Y}\right]$$
$$= \mu_y \tag{7}$$

The variance of $\tilde{Y}$ is equal to σ_y^2, which also did not change from Part I. This proof requires the results of the expectations calculated in Appendix equations A and B.

$$\begin{aligned} variance &= \mathbb{E}\Big[\tilde{Y}^2\Big] - \Big[\mathbb{E}\Big[\tilde{Y}\Big]\Big]^2 \\ &= \sigma_y^2 + \mu_y^2 - \mu_y^2 \\ &= \sigma_y^2 \end{aligned} \tag{8}$$

The correlation of $\tilde{X}$ and $\tilde{Y}$ did change from Part I as it is now negative. This proof requires the results of the expectations calculated in Appendix equations A, B and C.

$$\begin{aligned} Correl &= \frac{\mathbb{E}[\tilde{X}\tilde{Y}] - \mathbb{E}[\tilde{X}]\mathbb{E}[\tilde{Y}]}{\sigma_x \sigma_y} \\ &= \frac{-\sigma_x \sigma_y \hat{\rho}_{xy} + \mu_x \mu_y - \mu_x \mu_y}{\sigma_x \sigma_y} \\ &= -\hat{\rho}_{xy} \end{aligned} \tag{9}$$

Appendix

A) The expected value of $\tilde{Y}$ is...

$$\begin{aligned} \mathbb{E}\Big[\tilde{Y}\Big] &= \mathbb{E}\Big[- \sigma_y \sqrt{\hat{\rho}_{xy}}\, Z_c - \sigma_y \sqrt{1-\hat{\rho}_{xy}}\, Z_y + \mu_y \Big] \\ &= \mathbb{E}\Big[- \sigma_y \sqrt{\hat{\rho}_{xy}}\, Z_c \Big] + \mathbb{E}\Big[- \sigma_y \sqrt{1-\hat{\rho}_{xy}}\, Z_y \Big] + \mathbb{E}\Big[\mu_y\Big] \\ &= -\sigma_y \sqrt{\hat{\rho}_{xy}}\, \mathbb{E}\Big[Z_c\Big] - \sigma_y \sqrt{1-\hat{\rho}_{xy}}\, \mathbb{E}\Big[Z_y\Big] + \mathbb{E}\Big[\mu_y\Big] \\ &= \mu_y \end{aligned} \tag{10}$$

B) The expected value of the square of $\tilde{Y}$ is...

$$\begin{aligned} \mathbb{E}\Big[\tilde{Y}^2\Big] &= \mathbb{E}\Big[\Big\{ - \sigma_y \sqrt{\hat{\rho}_{xy}}\, Z_c - \sigma_y \sqrt{1-\hat{\rho}_{xy}}\, Z_y + \mu_y \Big\}^2\Big] \\ &= \mathbb{E}\Big[\sigma_y^2 \hat{\rho}_{xy} Z_c^2 + \sigma_y^2 (1-\hat{\rho}_{xy}) Z_y^2 + 2\sigma_y^2 \sqrt{\hat{\rho}_{xy}}\sqrt{1-\hat{\rho}_{xy}} Z_c Z_y - 2\mu_y \sigma_y \sqrt{\hat{\rho}_{xy}} Z_c - 2\mu_y \sigma_y \sqrt{1-\hat{\rho}_{xy}} Z_y + \mu_y^2\Big] \\ &= \mathbb{E}\Big[\sigma_y^2 \hat{\rho}_{xy} Z_c^2\Big] + \mathbb{E}\Big[\sigma_y^2 (1-\hat{\rho}_{xy}) Z_y^2\Big] + \mathbb{E}\Big[2\sigma_y^2 \sqrt{\hat{\rho}_{xy}}\sqrt{1-\hat{\rho}_{xy}} Z_c Z_y\Big] + \mathbb{E}\Big[- 2\mu_y \sigma_y \sqrt{\hat{\rho}_{xy}} Z_c\Big] \\ &+ \mathbb{E}\Big[- 2\mu_y \sigma_y \sqrt{1-\hat{\rho}_{xy}} Z_y\Big] + \mathbb{E}\Big[\mu_y^2\Big] \\ &= \sigma_y^2 \hat{\rho}_{xy}\, \mathbb{E}\Big[Z_c^2\Big] + \sigma_y^2 (1-\hat{\rho}_{xy})\, \mathbb{E}\Big[Z_y^2\Big] + 2\sigma_y^2 \sqrt{\hat{\rho}_{xy}}\sqrt{1-\hat{\rho}_{xy}}\, \mathbb{E}\Big[Z_c Z_y\Big] - 2\mu_y \sigma_y \sqrt{\hat{\rho}_{xy}}\, \mathbb{E}\Big[Z_c\Big] \\ &- 2\mu_y \sigma_y \sqrt{1-\hat{\rho}_{xy}}\, \mathbb{E}\Big[Z_y\Big] + \mathbb{E}\Big[\mu_y^2\Big] \\ &= \sigma_y^2 \hat{\rho}_{xy} + \sigma_y^2 (1-\hat{\rho}_{xy}) + \mu_y^2 \\ &= \sigma_y^2 + \mu_y^2 \end{aligned} \tag{11}$$

C) The expected value of the product of $\tilde{X}$ and $\tilde{Y}$ is...

$$
\begin{aligned}
\mathbb{E}\Big[\tilde{X}\tilde{Y}\Big] &= \mathbb{E}\Bigg[\Bigg\{\sigma_x\sqrt{\hat{\rho}_{xy}}\,Z_c + \sigma_x\sqrt{1-\hat{\rho}_{xy}}\,Z_x + \mu_x\Bigg\}\Bigg\{-\sigma_y\sqrt{\hat{\rho}_{xy}}\,Z_c - \sigma_y\sqrt{1-\hat{\rho}_{xy}}\,Z_y + \mu_y\Bigg\}\Bigg] \\
&= \mathbb{E}\Bigg[-\sigma_x\sigma_y\hat{\rho}_{xy}\,Z_c^2 - \sigma_x\sigma_y\sqrt{\hat{\rho}_{xy}}\sqrt{1-\hat{\rho}_{xy}}\,Z_cZ_y + \mu_y\sigma_x\sqrt{\hat{\rho}_{xy}}\,Z_c - \sigma_x\sigma_y\sqrt{\hat{\rho}_{xy}}\sqrt{1-\hat{\rho}_{xy}}\,Z_cZ_x \\
&\quad - \sigma_x\sigma_y(1-\hat{\rho}_{xy})\,Z_xZ_y + \mu_y\sigma_x\sqrt{1-\hat{\rho}_{xy}}\,Z_x - \mu_x\sigma_y\sqrt{\hat{\rho}_{xy}}\,Z_c - \mu_x\sigma_y\sqrt{1-\hat{\rho}_{xy}}\,Z_y + \mu_x\mu_y\Bigg] \\
&= \mathbb{E}\Big[-\sigma_x\sigma_y\hat{\rho}_{xy}\,Z_c^2\Big] + \mathbb{E}\Big[-\sigma_x\sigma_y\sqrt{\hat{\rho}_{xy}}\sqrt{1-\hat{\rho}_{xy}}\,Z_cZ_y\Big] + \mathbb{E}\Big[\mu_y\sigma_x\sqrt{\hat{\rho}_{xy}}\,Z_c\Big] + \mathbb{E}\Big[-\sigma_x\sigma_y\sqrt{\hat{\rho}_{xy}}\sqrt{1-\hat{\rho}_{xy}}\,Z_cZ_x\Big] \\
&\quad + \mathbb{E}\Big[-\sigma_x\sigma_y(1-\hat{\rho}_{xy})\,Z_xZ_y\Big] + \mathbb{E}\Big[\mu_y\sigma_x\sqrt{1-\hat{\rho}_{xy}}\,Z_x\Big] + \mathbb{E}\Big[-\mu_x\sigma_y\sqrt{\hat{\rho}_{xy}}\,Z_c\Big] + \mathbb{E}\Big[-\mu_x\sigma_y\sqrt{1-\hat{\rho}_{xy}}\,Z_y\Big] + \mathbb{E}\Big[\mu_x\mu_y\Big] \\
&= -\sigma_x\sigma_y\hat{\rho}_{xy}\,\mathbb{E}\Big[Z_c^2\Big] - \sigma_x\sigma_y\sqrt{\hat{\rho}_{xy}}\sqrt{1-\hat{\rho}_{xy}}\,\mathbb{E}\Big[Z_cZ_y\Big] + \mu_y\sigma_x\sqrt{\hat{\rho}_{xy}}\,\mathbb{E}\Big[Z_c\Big] - \sigma_x\sigma_y\sqrt{\hat{\rho}_{xy}}\sqrt{1-\hat{\rho}_{xy}}\,\mathbb{E}\Big[Z_cZ_x\Big] \\
&\quad - \sigma_x\sigma_y(1-\hat{\rho}_{xy})\,\mathbb{E}\Big[Z_xZ_y\Big] + \mu_y\sigma_x\sqrt{1-\hat{\rho}_{xy}}\,\mathbb{E}\Big[Z_x\Big] - \mu_x\sigma_y\sqrt{\hat{\rho}_{xy}}\,\mathbb{E}\Big[Z_c\Big] - \mu_x\sigma_y\sqrt{1-\hat{\rho}_{xy}}\,\mathbb{E}\Big[Z_y\Big] + \mathbb{E}\Big[\mu_x\mu_y\Big] \\
&= -\sigma_x\sigma_y\hat{\rho}_{xy} + \mu_x\mu_y
\end{aligned}
\tag{12}
$$